Exercise Set 1.1

Appendix B contains either full or partial solutions to all exercises with blue numbers. When the solution is not complete, the exercise number has an # next to it. A * next to an exercise number signals that the exercise is more challenging than usual. Be careful not to get into the habit of turning to the solutions too quickly. Make every effort to work exercises on your own before checking your answers. See the Preface for additional sources of assistance and further study.

In each of 1–6, fill in the blanks using a variable or variables to rewrite the given statement.

1. Is there a real number whose square is −1?
   a. Is there a real number x such that ______?
   b. Does there exist ______ such that $x^2 = -1$?

2. Is there an integer that has a remainder of 2 when it is divided by 5 and a remainder of 3 when it is divided by 6?
   a. Is there an integer n such that n has ______?
   b. Does there exist ______ such that if n is divided by 5 the remainder is 2 and if ______?
   Note: There are integers with this property. Can you think of one?

3. Given any two real numbers, there is a real number in between.
   a. Given any two real numbers $a$ and $b$, there is a real number $c$ such that ______
   b. For any two ______ such that $a < c < b$.

4. Given any real number, there is a real number that is greater.
   a. Given any real number $r$, there is ______ such that $s$ is ______
   b. For any ______, ______ such that $s > r$.

5. The reciprocal of any positive real number is positive.
   a. Given any positive real number $r$, the reciprocal of ______
   b. For any real number $r$, if $r$ is ______, then ______
   c. If a real number $r ______$, then ______

6. The cube root of any negative real number is negative.
   a. Given any negative real number $s$, the cube root of ______
   b. For any real number $s$, if $s$ is ______, then ______
   c. If a real number $s ______$, then ______

7. Rewrite the following statements less formally, without using variables. Determine, as best as you can, whether the statements are true or false.
   a. There are real numbers $u$ and $v$ with the property that $u + v < u - v$.
   b. There is a real number $x$ such that $x^2 < x$.
   c. For all positive integers $n$, $n^2 \geq n$.
   d. For all real numbers $a$ and $b$, $|a + b| \leq |a| + |b|$.

In each of 8–13, fill in the blanks to rewrite the given statement.

8. For all objects $J$, if $J$ is a square then $J$ has four sides.
   a. All squares ______
   b. Every square ______
   c. If an object is a square, then it ______

9. For all equations $E$, if $E$ is quadratic then $E$ has at most two real solutions.
   a. All quadratic equations ______
   b. Every quadratic equation ______
   c. If an equation is quadratic, then it ______
   d. If $E ______$, then $E ______$
   e. For all quadratic equations $E$, ______

10. Every nonzero real number has a reciprocal.
    a. All nonzero real numbers ______
    b. For all nonzero real numbers $r$, there is ______ for $r$.
    c. For all nonzero real numbers $r$, there is a real number $s$ such that ______

11. Every positive number has a positive square root.
    a. All positive numbers ______
    b. For any positive number $x$, there is ______ for $r$.
    c. For all positive numbers $x$, there is a positive number $r$ such that ______

12. There is a real number whose product with every number leaves the number unchanged.
    a. Some ______ has the property that its ______
    b. There is a real number $r$ such that the product of ______
    c. There is a real number $r$ with the property that for every real number $x ______$

13. There is a real number whose product with every real number equals zero.
    a. Some ______ has the property that its ______
    b. There is a real number $a$ such that the product of a ______
    c. There is a real number $a$ with the property that for every real number $b$, ______
Test Yourself

1. When the elements of a set are given using the set-roster notation, the order in which they are listed _____.
2. The symbol $\mathbb{R}$ denotes _____.
3. The symbol $\mathbb{Z}$ denotes _____.
4. The symbol $\mathbb{Q}$ denotes _____.
5. The notation $\{x \mid P(x)\}$ is read _____.
6. For a set $A$ to be a subset of a set $B$ means that, _____.
7. Given sets $A$ and $B$, the Cartesian product $A \times B$ is _____.

Answers for Test Yourself

1. does not matter
2. the set of all real numbers
3. the set of all integers
4. the set of all rational numbers
5. the set of all $x$ such that $P(x)$
6. every element in $A$ is an element in $B$
7. the set of all ordered pairs $(a, b)$ where $a$ is in $A$ and $b$ is in $B$

Exercise Set 1.2

1. Which of the following sets are equal?

   $A = \{a, b, c, d\}$
   $B = \{d, e, a, c\}$
   $C = \{d, b, a, c\}$
   $D = \{a, d, c, e\}$

2. Write in words how to read each of the following cutouts:

   a. $\{x \in \mathbb{R} \mid 0 < x < 1\}$
   b. $\{x \in \mathbb{R} \mid x \leq 0 \text{ or } x < 1\}$
   c. $\{n \in \mathbb{Z} \mid n \text{ is a factor of 6}\}$
   d. $\{n \in \mathbb{Z}^- \mid n \text{ is a factor of 6}\}$

3. a. Is $4 = \{4\}$?
   b. How many elements are in the set $\{3, 4, 3, 5\}$?
   c. How many elements are in the set $\{1, \{1\}, \{1, 1\}\}$?

4. a. Is $2 \in \{2\}$?
   b. How many elements are in the set $\{2, 2, 2, 2\}$?
   c. How many elements are in the set $\{0, \{0\}\}$?
   d. Is $0 \in \{\{0\}, \{1\}\}$?

5. Which of the following sets are equal?

   $A = \{0, 1, 2\}$
   $B = \{x \in \mathbb{R} \mid -1 \leq x < 3\}$
   $C = \{x \in \mathbb{R} \mid -1 < x < 3\}$
   $D = \{x \in \mathbb{Z} \mid -1 < x < 3\}$
   $E = \{x \in \mathbb{Z}^+ \mid 1 \leq x < 3\}$

6. For each integer $n$, let $T_n = \{n, n^2\}$. How many elements are in each of $T_2, T_3, T_4$ and $T_6$? Justify your answers.

7. Use the set-roster notation to indicate the elements in each of the following sets:

   a. $S = \{n \in \mathbb{Z} \mid n = (-1)^k \text{, for some integer } k\}$
   b. $T = \{n \in \mathbb{Z} \mid n = 1 + (-1)^k \text{, for some integer } k\}$

8. Let $A = \{c, d, f, g\}$, $B = \{f, j\}$, and $C = \{d, g\}$.

   Answer each of the following questions. Give reasons for your answer.

   a. Is $B \subseteq A$?
   b. Is $C \subseteq A$?
   c. Is $C \subseteq C$?
   d. Is $C$ a proper subset of $A$?

9. a. Is $3 \in \{1, 2, 3\}$?
   b. Is $1 \subseteq \{1\}$?
   c. Is $\{2\} \in \{1, 2\}$?
   d. Is $\{3\} \in \{1, 2, 3\}$?
   e. Is $1 \in \{1\}$?
   f. Is $\{2\} \subseteq \{1, 2, 3\}$?
   g. Is $\{1\} \subseteq \{1, 2\}$?
   h. Is $1 \in \{\{1\}, 2\}$?
   i. Is $\{\{1\}, 2\} \subseteq \{1, 2\}$?
   j. Is $\{1\} \subseteq \{1\}$?

10. a. Is $(-2)^2 = \sqrt{-2}$?
    b. Is $(5, -5) = (-5, 5)$?
    c. Is $[8 - 9, \sqrt{-1}] = (-1, -1)$?
    d. Is $\left(\frac{3}{2}, (-2)^0\right) = \left(\frac{3}{2}, -1\right)$?

11. Let $A = \{x, y, z\}$ and $B = \{a, b\}$. Use the set-roster notation to write each of the following sets, and indicate the number of elements that are in each set:

    a. $A \times B$
    b. $B \times A$
    c. $A \times A$
    d. $B \times B$

12. Let $S = \{2, 4, 6\}$ and $T = \{1, 3, 5\}$. Use the set-roster notation to write each of the following sets, and indicate the number of elements that are in each set:

    a. $S \times T$
    b. $T \times S$
    c. $S \times S$
    d. $T \times T$
Test Yourself

1. Given sets $A$ and $B$, a relation from $A$ to $B$ is ____.

2. A function $F$ from $A$ to $B$ is a relation from $A$ to $B$ that satisfies the following two properties:
   a. for every element $x$ of $A$, there is ____
   b. for all elements $x$ in $A$ and $y$ and $z$ in $B$, if ____ then ____

3. If $F$ is a function from $A$ to $B$ and $x$ is an element of $A$, then $F(x)$ is ____

Answers for Test Yourself

1. a subset of the Cartesian product $A \times B$ 2. a. an element $y$ of $B$ such that $(x, y) \in F$ (i.e., such that $x$ is related to $y$ by $F$) b. $(x, y) \in F$ and $(x, z) \in F$, $y = z$ 3. the unique element of $B$ that is related to $x$ by $F$

Exercise Set 1.3

1. Let $A = \{2, 3, 4\}$ and $B = \{6, 8, 10\}$ and define a relation $R$ from $A$ to $B$ as follows: For all $(x, y) \in A \times B$,
   
   $(x, y) \in R$ means that $\frac{y}{x}$ is an integer.

   a. Is $4 R 6$? Is $4 R 8$? Is $3, 8 \in R$? Is $(2, 10) \in R$?
   b. Write $R$ as a set of ordered pairs.
   c. Write the domain and co-domain of $R$.
   d. Draw an arrow diagram for $R$.

2. Let $C = D = \{-3, -2, -1, 1, 2, 3\}$ and define a relation $S$ from $C$ to $D$ as follows: For all $(x, y) \in C \times D$,
   
   $(x, y) \in S$ means that $\frac{x}{y}$ is an integer.

   a. Is $2 S 2$? Is $-1 S 1$? Is $(3, 3) \in S$? Is $(3, -3) \in S$?
   b. Write $S$ as a set of ordered pairs.
   c. Write the domain and co-domain of $S$.
   d. Draw an arrow diagram for $S$.

3. Let $E = \{1, 2, 3\}$ and $F = \{-2, -1, 0\}$ and define a relation $T$ from $E$ to $F$ as follows: For all $(x, y) \in E \times F$,
   
   $(x, y) \in T$ means that $\frac{x - y}{3}$ is an integer.

   a. Is $3 T 0$? Is $1 T (-1)$? Is $(2, -1) \in T$? Is $(3, -2) \in T$?
   b. Write $T$ as a set of ordered pairs.
   c. Write the domain and co-domain of $T$.
   d. Draw an arrow diagram for $T$.

4. Let $G = \{-2, 0, 2\}$ and $H = \{4, 6, 8\}$ and define a relation $V$ from $G$ to $H$ as follows: For all $(x, y) \in G \times H$,
   
   $(x, y) \in V$ means that $\frac{x - y}{4}$ is an integer.

   a. Is $2 V 6$? Is $(-2) V (-6)$? Is $(0, 6) \in V$? Is $(2, 4) \in V$?
   b. Write $V$ as a set of ordered pairs.
   c. Write the domain and co-domain of $V$.
   d. Draw an arrow diagram for $V$.

5. Define a relation $S$ from $R$ to $R$ as follows:
   
   For all $(x, y) \in R \times R$,
   
   $(x, y) \in S$ means that $x \geq y$.

   a. Is $(2, 1) \in S$? Is $(2, 2) \in S$? Is $(2, 3) \in S$? Is $(-1) S (2)$?
   b. Draw the graph of $S$ in the Cartesian plane.

6. Define a relation $R$ from $R$ to $R$ as follows:
   
   For all $(x, y) \in R \times R$,
   
   $(x, y) \in R$ means that $y = x^2$.

   a. Is $(2, 4) \in R$? Is $(4, 2) \in R$? Is $(2, 3) \in R$? Is $9 R (-3)$?
   b. Draw the graph of $R$ in the Cartesian plane.

7. Let $A = \{4, 5, 6\}$ and $B = \{5, 6, 7\}$ and define relations $R$, $S$, and $T$ from $A$ to $B$ as follows:
   
   For all $(x, y) \in A \times B$,
   
   $(x, y) \in R$ means that $x \geq y$.

   a. Write $R$ as a set of ordered pairs.
   b. Indicate whether any of the relations $R$, $S$, and $T$ are functions.

8. Let $A = \{2, 4\}$ and $B = \{1, 3, 5\}$ and define relations $U$, $V$ and $W$ from $A$ to $B$ as follows: For all $(x, y) \in A \times B$,
   
   $(x, y) \in U$ means that $y - x > 2$.
   $(x, y) \in V$ means that $y - 1 = \frac{x}{2}$.
   $W = \{(2, 5), (4, 1), (2, 3)\}$.
a. Draw arrow diagrams for \( U, V, \) and \( W. \)
b. Indicate whether any of the relations \( U, V, \) and \( W \) are functions.

9. a. Find all relations from \( \{0, 1\} \) to \( \{1\}. \)
b. Find all functions from \( \{0, 1\} \) to \( \{1\}. \)
c. What fraction of the relations from \( \{0, 1\} \) to \( \{1\} \) are functions?

10. Find four relations from \( \{a, b\} \) to \( \{x, y\} \) that are not functions from \( \{a, b\} \) to \( \{x, y\}. \)

11. Define a relation \( P \) from \( \mathbb{R}^+ \) to \( \mathbb{R} \) as follows: For all real numbers \( x \) and \( y \) with \( x > 0, \)
\[(x, y) \in P \text{ means that } x = y^2.\]
Is \( P \) a function? Explain.

12. Define a relation \( T \) from \( \mathbb{R} \) to \( \mathbb{R} \) as follows: For all real numbers \( x \) and \( y, \)
\[(x, y) \in T \text{ means that } y^2 - x^2 = 1.\]
Is \( T \) a function? Explain.

13. Let \( A = \{-1, 0, 1\} \) and \( B = \{a, b, u, v, w\}. \) Define a function \( F: A \to B \) by the following arrow diagram:

a. Write the domain and co-domain of \( F. \)
b. Find \( F(-1), F(0), \) and \( F(1). \)

14. Let \( C = \{1, 2, 3, 4\} \) and \( D = \{a, b, c, d\}. \) Define a function \( G: C \to D \) by the following arrow diagram:

a. Write the domain and co-domain of \( G. \)
b. Find \( G(1), G(2), G(3), \) and \( G(4). \)

15. Let \( X = \{2, 4, 5\} \) and \( Y = \{1, 2, 4, 6\}. \) Which of the following arrow diagrams determine functions from \( X \) to \( Y? \)

16. Let \( f \) be the squaring function defined in Example 1.3.6. Find \( f(-1), f(0), \) and \( f \left(\frac{1}{2}\right). \)

17. Let \( g \) be the successor function defined in Example 1.3.6. Find \( g(-1000), g(0), \) and \( g(999). \)

18. Let \( h \) be the constant function defined in Example 1.3.6. Find \( h \left(\frac{-12}{5}\right), h \left(\frac{2}{3}\right), \) and \( h \left(\frac{5}{7}\right). \)

19. Define functions \( f \) and \( g \) from \( \mathbb{R} \) to \( \mathbb{R} \) by the following formulas: For all \( x \in \mathbb{R}, \)
\[ f(x) = 2x \quad \text{and} \quad g(x) = \frac{2x^2 + 2x}{x^2 + 1}. \]
Does \( f = g? \) Explain.

20. Define functions \( H \) and \( K \) from \( \mathbb{R} \) to \( \mathbb{R} \) by the following formulas: For all \( x \in \mathbb{R}, \)
\[ H(x) = (x - 2)^2 \quad \text{and} \quad K(x) = (x - 1)(x - 3) + 1. \]
Does \( H = K? \) Explain.